J. Fluid Mech. (1985), vol. 156, pp. 387–399
Printed in Great Britain

Near-neutral centre-modes as inviscid perturbations to a trailing line vortex

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(Received 3 December 1984)

Inviscid linear perturbations to a columnar trailing line vortex are found in the form of centre-modes. These near-neutral modes, occurring at moderate values of the azimuthal wavenumber n, are the analogue of the ring modes for large n discussed by Stewartson & Capell (1985). The appearance and disappearance of these modes as the swirl parameter varies may partly explain the difficulties encountered by numerical analysts in the computation of such modes. In addition, instabilities are found at higher values of the swirl parameter than have previously been reported.

1. Introduction

The problem of the linear instability of an unbounded columnar vortex has recently attracted much interest, in the expectation that it will be relevant to the important and dramatic phenomenon of vortex breakdown. A specific such swirling flow often studied numerically is that of which the undisturbed flow has non-dimensional azimuthal and axial velocities V and W that are functions only of the distance r from the axis of the vortex and given by

$$V = \frac{q}{r}(1-e^{-r^2}), \quad W = e^{-r^2}, \tag{1.1}$$

where q is a constant and the inverse of a Rossby number. Numerical solutions for inviscid disturbances to this flow have been computed for various values of q by Lessen, Singh & Paillet (1974), by Duck & Foster (1980) and more recently by Leibovich & Stewartson (1983). If θ and z are the azimuthal and axial coordinates respectively, and if a weak time-dependent perturbation to each component of velocity and the pressure is assumed to be of the form

$$Q(r)\exp[i(\beta nz-n\theta-\omega t)], \tag{1.2}$$

where n and β are given constants, of which (without loss of generality) n is a positive integer, and ω is a complex constant to be found, the results of Lessen *et al.* indicate that the most unstable modes have $\beta q > 0$ and that the growth rate increases with n. Subsequent studies have confirmed this, and Leibovich & Stewartson present, in addition to computations at finite values of n, an asymptotic analysis for the unstable modes when $n \gg 1$ and a sufficient condition for instability for general V, W. For the flow given by (1.1) this condition reduces to $q^2 < 2$, though, as noted by previous authors and confirmed here, this is not necessary for instability at finite values of n.

Because of the practical importance of the determination of the value of q above which the vortex is stable, the neutral modes are of considerable interest, and it is noteworthy that the computation of these has been found extremely difficult. The asymptotic analysis of Leibovich & Stewartson (1983) was valid for

$$\tfrac{1}{2}q < \beta < q^{-1}, \tag{1.3}$$

and the authors conjectured that $\beta = q^{-1}$ and $\beta = \frac{1}{2}q$ were, in the limit $n \to \infty$, respectively the positions of the upper and lower neutral points, with the flow being stable for $q^2 > 2$ and n sufficiently large. Their analysis did not hold in the immediate neighbourhoods of these neutral points, but has subsequently been refined to do so in papers by Stewartson & Capell (1985) and Stewartson & Leibovich (1985). In the former the authors consider the neighbourhood of the upper neutral point given by $n^2(1-\beta q) = O(1)$ and find weakly unstable ring-modes with $\beta = q^{-1}$ as neutral point. As $n^2(1-\beta q) \to \infty$ there is a match between these ring-modes and those of Leibovich & Stewartson (1983). If $1-\beta q < 0$ there are not expected to be any unstable modes for any n. As $q^2 \to 2^-$, the value of r on which the ring-mode is centred tends to zero. In the second paper the neighbourhood of $\beta = \frac{1}{2}q$ is discussed. In this case the disturbance is concentrated near the axis of the vortex, and the main contribution to the eigenvalue ω is found, when $n \gg 1$, by deforming the path of integration of the differential equation away from the real axis of r to go through a saddle-point in the complex r-plane. The neutral point is thus identified with $q-2\beta = O(n^{-\frac{2}{3}})$ when this saddle-point lies on the imaginary r-axis. In addition, the marginal stability of the flow is examined. This, in the limit $n \to \infty$, occurs at $q^2 = 2$, and it is shown that if $q^2 < 2$ the two neutral points occur at $\beta = \beta_1, \beta_2$, with

$$\beta_1 < \tfrac{1}{2}q < \frac{1}{\sqrt{2}} < q^{-1} = \beta_2$$

each of the quantities stated differing by $O(n^{-1})$, while if $q^2 > 2$ the corresponding result is

$$\beta_1 < \beta_2 < q^{-1} < \frac{1}{\sqrt{2}} < \tfrac{1}{2}q.$$

It is also shown that, when $n \gg 1$, the flow is stable if

$$q^2 > 2\left(1 + \frac{1}{n}\sqrt{\frac{2}{3}}\right).$$

In the present paper we examine the neighbourhood of the neutral point $\beta = q^{-1}$ for finite values of n. If $q^2 < 2$ this is the upper neutral point for modes whose lower neutral point is, in the limit $n \to \infty$, $\beta = \frac{1}{2}q$. If $q^2 > 2$ this again must be an upper neutral point, since for instability $\beta < q^{-1}$, though the corresponding lower neutral point cannot now be $\beta = \frac{1}{2}q$. The results, which turned out to be rather unexpected, may partially explain the difficulties experienced by numerical analysts in computing modes that are nearly neutral. The modes are centre-modes, and they are found to exist, for given n, not for all q below some upper bound, but for discrete intervals of q. For example, if $n = 1$ such modes are found for $0.21 \leqslant q^2 \leqslant \frac{2}{9}$ in the range $q^2 < 2$, and in the range $q^2 > 2$ for $5.33 \geqslant q^2 \geqslant 3.15$ and $2.64 \geqslant q^2 \geqslant 2.43$, and also in decreasing intervals centred on $q^2 = 2(4m^2-1)/(4m^2-9)$ for integral m, this last result having been verified for m from 4 to 20, but probably holding for all m. Similar results are found for $n = 2, 3$, though for $n = 4$, $q^2 = 2$ does not seem to a limit point for such modes. As n increases, the findings are not inconsistent with the results of Leibovich & Stewartson (1983), who predict stability for $q^2 > 2$ as $n \to \infty$, and of Stewartson & Capell (1985), who obtain a centre-mode, with which the present results might be expected to lead to a match, only in the limit $q^2 \to 2^-$.

The plan of the paper is as follows. In §2 we set up the relevant equation for the radial component of the perturbation velocity and explain the strategy of the analysis for determining these centre-modes when $0 < 1-\beta q \ll 1$. In §§3 and 4 we set up the equations that are valid respectively away from, and in the immediate neighbourhood of, the axis of the vortex and demonstrate how the match is to be made between them. This match determines the leading contribution to ω_i, the imaginary part of ω. The main results of the paper are contained in §5 in which the numerical work required to determine those values of q for which the modes exist is described and the conclusions are demonstrated graphically. In §6 we carry out a similar analysis for $0 < \beta \ll 1$, as it seemed a possibility that $\beta = 0$ could be a candidate for the position of the lower neutral point. However, the conclusion from this investigation is that it is not, at least for centre-modes of the type considered here.

2. The equation for the perturbed radial velocity and the strategy of the centre-modes

Non-dimensional coordinates and velocity components are defined as follows for this inviscid incompressible vortex flow. We take cylindrical polar coordinates (r, θ, z), with z along the axis of the vortex, and the basic flow to be steady and, for the trailing vortex to be considered here, to have corresponding velocity components $(0, V, W)$, where V and W are given by (1.1). In the disturbed flow each velocity component and the pressure are assumed to be of the form (1.2).

It has been shown by Howard & Gupta (1962) (see also Pedley 1968) that in the linearized equations of motion the pressure and axial and azimuthal velocity perturbations may be eliminated in favour of the radial perturbation $u\,\mathrm{e}^{\mathrm{i}(n\beta z-n\theta-\omega t)}$ to give the following equation for $y = ru(r)$:

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{ry'(r)}{1+\beta^2r^2}\right)-n^2\left(1+\frac{a(r)}{n\gamma(r)}+\frac{b(r)}{\gamma^2(r)}\right)y(r) = 0, \tag{2.1}$$

where $a(r)$, $b(r)$ and $\gamma(r)$ are as in (4.4) of Leibovich & Stewartson (1983) and their ϕ of equation (4.1) is related to y by $r^{\frac{1}{2}}y = (1+\beta^2r^2)^{\frac{1}{2}}\phi$. Thus

$$a(r) = \frac{4r^2\,\mathrm{e}^{-r^2}}{(1+\beta^2r^2)^2}[q+\beta^2q-\beta+(\beta+q\beta^2)\,r^2+\beta^3r^4], \tag{2.2a}$$

$$b(r) = \frac{4\beta q(1-\beta q)\,\mathrm{e}^{-r^2}(1-\mathrm{e}^{-r^2})}{1+\beta^2r^2}, \tag{2.2b}$$

$$\gamma(r) = n[\beta\,\mathrm{e}^{-r^2}-qr^{-2}(1-\mathrm{e}^{-r^2})]-\omega, \tag{2.2c}$$

and the boundary conditions to be satisfied by y in (2.1) are

$$y(0) = 0, \quad y(\infty) = 0. \tag{2.3}$$

As discussed in §1, we shall assume that $\beta q > 0$, as this has previously been found to lead to the most unstable modes, and henceforth, for clarity and convenience, will take $\beta > 0$, $q > 0$.

The types of modes to be sought here are centre-modes, so called because the disturbance is concentrated mainly in the neighbourhood of the axis of the vortex. These modes will be almost neutral, and as such will, to leading order, possess critical layers through which, for this inviscid normal-mode approach, the solution must be continued analytically. As indicated by the numerical work of previous authors discussed here in §1, and verified by asymptotic techniques by Stewartson & Capell (1985) for $n \gg 1$, the point $\beta q = 1$ is very probably neutral. We assume that

$$0 < 1-\beta q \ll 1, \tag{2.4}$$

and examine the possibility of the existence of centre-modes near $\beta = q^{-1}$ for values of n of order unity.

The procedure for the discussion of the centre-modes is as follows. It closely resembles that employed by Stewartson & Brown (1984) in their study of the centre-modes in rotating Hagen–Poiseuille flow. We first choose ω in (2.2c) so that, to leading order in the small parameter $1-\beta q$, $\gamma(0) = 0$, and thus set

$$\omega - n(\beta - q) = \frac{n}{2q}(q^2-2)\,\Gamma, \tag{2.5}$$

where Γ is a small complex constant to be found; it will emerge that $|\Gamma| = O(1-\beta q)$. Equation (2.1) is to be solved in two separate regions. When $r = O(1)$, i.e. away from the immediate neighbourhood of the axis of the vortex, we set $\beta q = 1$ and $\Gamma = 0$ in (2.1) and find, in §3, the solution of the resulting equation that decays at infinity. In this outer equation the term in $b(r)/\gamma^2$, the most singular term in (2.1), is absent because of the factor $1-\beta q$. Since, near $r = 0$, it follows from (2.2c) and (2.5) that

$$\gamma(r) \approx -\frac{n}{2q}(q^2-2)(\Gamma - r^2), \tag{2.6}$$

Γ must be retained when $r^2 = O(|\Gamma|)$, and this defines the order of magnitude of the size of the inner region to be discussed in §4. On the assumption that $|\Gamma| = O(1-\beta q)$, the term $b(r)/\gamma^2$ leads to a contribution to the inner equation that is of comparable magnitude to those of the other terms. The solution of the inner equation must vanish at the origin and match with the outer solution in the appropriate limits. It is this matching that leads to the value of Γ.

Since $b(r)$ in (2.2c) has, in addition, a factor β, a similar analysis may be carried through when $\beta \approx 0$ instead of $\beta \approx q^{-1}$. This we undertake in §6 and find that there are no centre-modes in this neighbourhood for any values of n or q. Had such modes existed this would have given support for $\beta = 0$ as a candidate for the position of the lower neutral point at finite values of n.

In §3 we establish the equation to be solved away from the immediate neighbourhood of $r = 0$ when (2.4) holds.

3. The solution when $r = O(1)$

As outlined in §2, for the solution away from the axis of the vortex we set $\beta q = 1$ and $\Gamma = 0$ in (2.1), which becomes

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{ry'}{r^2+q^2}\right) - \left(\frac{n^2}{q^2} + \frac{4r^4\,\mathrm{e}^{-r^2}}{q^2(r^2-1+\mathrm{e}^{-r^2}) - r^2(1-\mathrm{e}^{-r^2})}\right) y = 0. \tag{3.1}$$

Of the two solutions of (3.1) we require the one that is exponentially small with exponent $-nr/q$ as $r \to \infty$, and for the match with the inner solution of §4 the only property we shall require of the solution is the ratio A/B when it was written as a sum of power series about the origin, i.e.

$$y = Ar^p \sum_{m=1}^{\infty} a_m r^{2(m-1)} + Br^{-p} \sum_{m=1}^{\infty} b_m r^{2(m-1)}. \tag{3.2}$$

Here $a_1 = b_1 = 1$, and $p(>0)$ is defined by

$$p^2 = n^2 + \frac{8q^2}{q^2-2}, \tag{3.3}$$

the origin being a regular singular point if $q^2 \neq 2$.

The determination of A/B is a numerical undertaking. It emerges that for the validity of the ensuing analysis we must have p real, so that, from (3.3), for every n there are two possible ranges of q^2, namely $q^2 > 2$ and $0 \leqslant q^2 \leqslant 2n^2/(n^2+8)$, the former corresponding to lower Rossby numbers. The intervening values of q^2 do not admit modes of the structure studied here. For any given q^2 with $p^2 > 0$ we proceeded as follows.

We first set

$$r^2 = x, \quad y(r) = Y(x) \tag{3.4}$$

in (3.1), so that it becomes

$$x(q^2+x)\,Y'' + q^2 Y' - x(q^2+x)^2 \left(\frac{n^2}{4q^2x^2} + \frac{\mathrm{e}^{-x}}{q^2(x-1+\mathrm{e}^{-x}) - x(1-\mathrm{e}^{-x})}\right) Y = 0, \tag{3.5}$$

and, for values of n and q^2 in (3.3) such that p was *not* an integer, (3.5) was integrated by a Runge–Kutta routine inwards from some large value of x, where Y was taken to be say, 10^{-5}, with $Y' = -nY/(2qx^{\frac{1}{2}})$ there, as far as $x = x_0$, where x_0 is some number less than the radius of convergence of the series solutions about $x = 0$ of (3.5). The radius of convergence of these solutions is either q^2 or $|\tilde{z}_1|$, where $\tilde{z}_1$ is the zero of $\{q^2(\tilde{z}-1+\mathrm{e}^{-\tilde{z}}) - \tilde{z}(1-\mathrm{e}^{-\tilde{z}})\}/\tilde{z}^2$ that is closest to the origin, whichever is the least. It is not difficult to show that $|\tilde{z}_1| \geqslant 2\pi$ when $q^2 = 0, 1, \infty$ and that $\tilde{z}_1 \to 0$ as $q^2 \to 2$. For the intermediate values of q^2 considered, the position of the zeros was examined numerically to ensure that the chosen x_0 was sufficiently small. Thus at $x = x_0$ we have the required solution together with its derivative. The next task was to evaluate the series solutions of (3.5), namely

$$x^{\frac{1}{2}p} \sum_{m=1}^{\infty} a_m x^{m-1} \quad \text{and} \quad x^{-\frac{1}{2}p} \sum_{m=1}^{\infty} b_m x^{m-1}, \tag{3.6}$$

and their derivatives, and to find A/B in (3.2) by equating a combination of the two series so that $Y(x_0)$ and $Y'(x_0)$ agreed with the values found for the corresponding quantities yielded by the Runge–Kutta procedure. For each value of q^2, A/B was calculated using two different choices of x_0 as a check that x_0 was well inside the circle of convergence and that the equation was being integrated sufficiently accurately between these two points. To find a_m and b_m we wrote (3.5) in the form

$$x^2 Y'' \sum_{m=1}^{\infty} \alpha_m x^{m-1} + xY' \sum_{m=1}^{\infty} \beta_m x^{m-1} - Y \sum_{m=1}^{\infty} \gamma_m x^{m-1} = 0, \tag{3.7}$$

where $\alpha_1 = \frac{1}{2}q^4 - q^2, \quad \alpha_m = \dfrac{(-1)^{m-1}}{(m+1)!}[q^4 - 2q^2(m+1) + m(m+1)] \quad (m \geqslant 2), \qquad (3.8a)$

$$\beta_m = \frac{(-1)^{m-1} q^2}{(m+1)!}(q^2 - m - 1) \quad (m \geqslant 1), \tag{3.8b}$$

and $\gamma_1 = q^4\delta_1, \quad \gamma_2 = q^4\delta_2 + 2q^2\delta_1, \quad \gamma_m = q^4\delta_m + 2q^2\delta_{m-1} + \delta_{m-2} \quad (m \geqslant 3),$

with $\delta_m = \dfrac{(-1)^{m-1}}{(m+1)!}\left[\dfrac{n^2}{4} - \dfrac{n^2(m+1)}{4q^2} + m(m+1)\right] \quad (m \geqslant 1). \qquad (3.8c)$

The coefficients a_m and b_m could then be determined successively from the triangular system $\boldsymbol{Cv} = 0$ where C_{ij}, the ijth element of the matrix $\boldsymbol{C}$, is given by

$$C_{ij} = (c+j-1)(c+j-2)\alpha_{i-j+1} + (c+j-1)\beta_{i-j+1} - \gamma_{i-j+1}, \tag{3.9}$$

where $c = \frac{1}{2}p$ if the vector $\boldsymbol{v}$ consists of the coefficients a_j and $c = -\frac{1}{2}p$ if it consists of b_j. There was no difficulty in obtaining as many (say 120) of a_j and b_j as required.

For the region $0 \leqslant q^2 \leqslant 2n^2/(n^2+8)$ an identical procedure was adopted after the equation had been scaled by the transformation $x = q^2\bar{x}$ so that the size of the corresponding coefficients $\bar{\alpha}_m$, β_m and $\bar{\gamma}_m$ could be kept under control when q^2 was very small.

When p in (3.3) is a positive integer, P say, b_{P+1} in (3.2) is infinite, an indication that the series should contain a term in $r^P \log r$. For our purpose it is sufficient to consider integral values of p as the limits $p \rightarrow P$ and note that in such situations we shall have $b_{P+1} = \bar{b}(p-P)^{-1}$, where $\bar{b}$ is finite. To cancel this singularity A/B must have a simple pole as $p \rightarrow P$, with residue L_P say, so that

$$\frac{A}{B} \approx \frac{L_P}{p-P} \quad \text{as } p \rightarrow P, \tag{3.10}$$

where $L_P = -\bar{b}$, and this property of A/B will be sufficient for our requirements. It is simple, for given q^2 and n, to calculate $\bar{b}$ the value of which is

$$\bar{b} = (\beta_1 P)^{-1} \sum_{j=1}^{P} C_{P+1,j} b_j \tag{3.11}$$

where the C_{ij}, and hence b_j, are calculated with c in (3.9) set equal to $-\frac{1}{2}P$. When $P = 0$ the corresponding result is that $A/B \rightarrow -1$ as $p \rightarrow 0$.

Before presenting results for selected values of n, we discuss the solution that is appropriate in the immediate neighbourhood of $r = 0$ so that the relevance of the quantity A/B becomes clear.

4. The solution in the immediate neighbourhood of the axis of the vortex

The procedure indicated in §2 suggests that we set

$$s = r^2/\Gamma \tag{4.1}$$

in (2.1), regard $|\Gamma|$ as $O(1-\beta q)$ and set $\beta q = 1$ except in the multiplying factor of $b(r)$. Thus, in the region where $s = O(1)$, (2.1) reduces to

$$s^2 \frac{\mathrm{d}^2\tilde{Y}}{\mathrm{d}s^2} + s\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}s} + \left(-\frac{n^2}{4} + \frac{2q^2 s}{(q^2-2)(1-s)} - \frac{4(1-\beta q)q^2 s}{\Gamma(q^2-2)^2(1-s)^2}\right)\tilde{Y} = 0, \tag{4.2}$$

where $\tilde{Y}(s) = y(r)$, the solution of which that is regular at $s = 0$ is

$$\tilde{Y}(s) = \frac{s^{\frac{1}{2}n}}{(1-s)^{\mu}} F(a, b, n+1, s) \tag{4.3}$$

if we choose

$$\mu(\mu+1) = \frac{4(1-\beta q)q^2}{\Gamma(q^2-2)^2}. \tag{4.4}$$

In (4.3), F is a hypergeometric function in the usual notation, with

$$a = \tfrac{1}{2}(n-p)-\mu, \quad b = \tfrac{1}{2}(n+p)-\mu, \tag{4.5}$$

where p is as in (3.3). To determine μ, and hence Γ, we must match this solution, as $s \rightarrow \infty$, with the solution of §3 that takes the form (3.2) as $r \rightarrow 0$. Now as $s \rightarrow \infty$

it follows from (4.3) and the properties of the hypergeometric function that, on replacing s by r^2/Γ,

$$y(r) \propto \frac{r^n}{(\Gamma - r^2)^\mu}\left[c_0\left(-\frac{r^2}{\Gamma}\right)^{\mu-\frac{1}{2}n+\frac{1}{2}p}\Sigma_1 + d_0\left(-\frac{r^2}{\Gamma}\right)^{\mu-\frac{1}{2}n-\frac{1}{2}p}\Sigma_2\right], \tag{4.6}$$

where Σ_1 and Σ_2 are power series in Γ/r^2 whose leading terms are both unity, $|\arg(-r^2/\Gamma)| < \pi$, and

$$c_0 = \frac{n!\,(p-1)!}{(\frac{1}{2}n+\frac{1}{2}p-\mu-1)!\,(\frac{1}{2}n+\frac{1}{2}p+\mu)!}, \quad d_0 = \frac{n!\,(-p-1)!}{(\frac{1}{2}n-\frac{1}{2}p-\mu-1)!\,(\frac{1}{2}n-\frac{1}{2}p+\mu)!}. \tag{4.7}$$

Now the leading terms of (3.2) give

$$y(r) \approx Ar^p + Br^{-p}, \tag{4.8}$$

of which the powers of r are seen to match automatically with those in (4.6). To match the coefficients, we require

$$\frac{A}{B} = \frac{c_0}{d_0}\left(-\frac{1}{\Gamma}\right)^p, \tag{4.9}$$

and since $p > 0$ and $|\Gamma| \ll 1$ we deduce that to leading order $c_0 = 0$. From (4.7) this implies that $\frac{1}{2}(n+p)-\mu$ is a negative integer or zero, i.e.

$$\mu = \tfrac{1}{2}(n+p)+M \ (= \mu_0 \text{ say}) \quad (M \geqslant 0) \tag{4.10}$$

for integral M, and F reduces to a polynomial in s of degree M. To be able to calculate the leading-order contribution to ω_i, the imaginary part of ω, we must, however, retain the next approximation to μ. On setting $\mu = \mu_0 + \Delta\mu$ in (4.7), where $|\Delta\mu| \ll 1$, we obtain

$$c_0 = (-1)^{M+1}\frac{n!\,M!\,(p-1)!}{(p+n+M)!}\Delta\mu, \quad d_0 = (-1)^M\frac{n!(p+M)!}{p!(n+M)!} \tag{4.11}$$

and then the matching condition (4.9) gives

$$\Delta\mu = -\frac{A}{B}\left(-\frac{1}{\Gamma}\right)^{-p}\frac{(p+n+M)!\,(p+M)!}{p!(p-1)!\,M!\,(n+M)!}. \tag{4.12}$$

However, it is the calculation of Γ, and hence of ω, that is the purpose of this study. To leading order Γ is real ($= \Gamma_0$ say) and is obtained from (4.4) with μ set equal to μ_0. To order $|\Delta\mu|$, (4.4) becomes

$$\Gamma \approx \Gamma_0\left\{1 - \frac{2\mu_0+1}{\mu_0(\mu_0+1)}\Delta\mu\right\}, \tag{4.13}$$

and this, together with (4.12), enables ω_i to be found on proper interpretation of the factor $(-\Gamma^{-1})^{-p}$.

Suppose first that $\beta q > 1$. Then Γ in (4.4) is negative, as is s defined by (4.1). In this case there is no difficulty in extending (4.3) through $s = 1$, or equivalently in evaluating $(-\Gamma^{-1})^{-p}$ in (4.12) as $[4(\beta q-1)q^2/(q^2-2)^2\mu_0(\mu_0+1)]^p$. Thus the contribution to Γ, and hence to ω, that we have calculated through (4.12) and (4.13) is purely real, and if $p > 1$ will be of higher order than that due to the terms of (2.1) that we have ignored. In this situation it seems that the solution of (2.1) may be extended to all powers of $\beta q - 1$, and at no stage will complex quantities be generated. The modes will be neutral without a critical layer, and may extend to large values of βq. However, when $\Gamma > 0$, $\tilde{Y}(s)$ in (4.3) has a branch point at $s = 1$, or equivalently at $r^2 = \Gamma$. The continuation of the solution through $r^2 = \Gamma$ is made by assuming that

Γ has a small positive imaginary part. This will, by (2.5), also give $\omega/(q^2-2)$ a small positive imaginary part, and the consistency of the assumption must be verified *a posteriori*. Since, from (4.6), we have for r^2 real and positive that $|\arg(-\Gamma^{-1}) < \pi|$, and $-\Gamma^{-1}$ will, like Γ, have a small positive imaginary part, it follows that we may write

$$(-\Gamma^{-1})^{-p} = |\Gamma_0|^p \, e^{-ip\pi} \tag{4.14}$$

in (4.12). Thus altogether, from (2.5), (4.12) and (4.13),

$$\omega - n(\beta - q) \approx \frac{2(1-\beta q)\, qn}{(q^2-2)\,\mu_0(\mu_0+1)} \left\{1 + \frac{A}{B}\frac{(p+n+M)!\,(p+M)!}{p!\,(p-1)!\,M!\,(n+M)!} \times \left(\frac{4(1-\beta q)\, q^2}{(q^2-2)^2\mu_0(\mu_0+1)}\right)^p \frac{2\mu_0+1}{\mu_0(\mu_0+1)}\, e^{-ip\pi}\right\}. \tag{4.15}$$

The consistency condition is that

$$\frac{A}{B}\sin p\pi < 0, \tag{4.16}$$

where A/B is calculated as described in §3. For those values of q^2 and n for which (4.16) is satisfied, we deduce that (2.1) possesses unstable centre-modes of the kind we have described when $\beta q \approx 1$. When (4.16) is not satisfied we conclude that there are no centre-modes in this neighbourhood. In §5 we examine the consistency condition for various values of n.

5. Specific results for various values of n

It was shown in §4 that a necessary condition for the existence of centre-modes of the type under discussion here is that (4.16) should hold. Here p is given in terms of q and n by (3.3), and A/B in (3.2) must be obtained from the solution of (3.5) that is exponentially small as $x \to \infty$. This is a simple numerical task, and was, for non-integral values of p, undertaken as described after (3.5). When p is an integer (P say), $(A/B)\sin p\pi$ may be determined as $(-1)^P \pi L_P$, where L_P is defined in (3.10).

For each n there are two ranges of q^2 to consider, namely $q^2 > 2$ and $0 \leqslant q^2 \leqslant 2n^2/(n^2+8)$. As $n = 1$ is likely to correspond to larger values of q^2 for which the flow is unstable, we consider it first. For the low integral values of p it is possible to calculate the corresponding $(A/B)\sin p\pi$ by hand, and specific results are that when $P = 0, 1, 3$ ($P = 2$ is not applicable with $n = 1$):

$$\frac{A}{B}\sin p\pi \approx -\sin p\pi \quad \text{as } p \to 0 \quad (q^2 = \tfrac{2}{9}), \tag{5.1a}$$

$$\frac{A}{B}\sin p\pi \approx \frac{1}{4q^2} \quad \text{as } p \to 1 \quad (q^2 \ll 1), \tag{5.1b}$$

$$\frac{A}{B}\sin p\pi \approx \tfrac{2}{45} \quad \text{as } p \to 3 \quad (q^2 \gg 1). \tag{5.1c}$$

We thus see immediately that there are no centre-modes for q^2 very small or very large, but that there are for $q^2 = (\frac{2}{9})^-$. It was also verified that $L_P < 0$ for $4 \leqslant P \leqslant 40$, from which it may be concluded that, at least for these values of P and probably for larger values as well, $(A/B)\sin p\pi$ is negative when p lies close to an even integer, but is positive when p lies close to an odd integer. Now as $q^2 \to 2^+$ the values of q^2

for which p is an integer have 2 as a limit point, and we thus have decreasing intervals in which the centre-modes exist. Also, as $P \to \infty$, L_P becomes exponentially large, and indeed the asymptotic form

$$\log |L_P| \approx 2P \log P - P \log 192 + O(1) \tag{5.2}$$

may be obtained by solving the equation

$$\frac{d^2 Y}{dz^2} + \frac{1}{z}\frac{dY}{dz} - \left(\frac{1}{4z^2} + \frac{4}{(q^2-2)(z^2+z^3)}\right) Y = 0, \tag{5.3}$$

which is obtained from (3.5) on setting $x = (q^2-2)z$ and letting $q^2 \to 2$ except in the last term. This equation yields a two-term recurrence relation, which may be solved exactly for the coefficients corresponding to b_j of (3.6) and the singularity in b_{P+1} analysed. Unfortunately this approximation did not yield the sign of L_P.

For the values of n and q^2 for which the result was specifically tested it so happened that A/B as a function of q^2 vanished at most once between values of q^2 that correspond to consecutive integral values of p. Thus $(A/B) \sin p\pi$, which is easy to compute at integral values of p, remained one signed for $P < p < P+1$ if it had the same sign for both $p = P$ and $p = P+1$, and vanished once otherwise. This result, which has not been proved to be general, simplified the computation of the values of $(A/B) \sin p\pi$, and for $n = 1$ these are illustrated in figure 1. Centre-modes exist for $(A/B) \sin p\pi < 0$ only, and it is seen that there are none for $q^2 > 5.33$, but they exist for $5.33 \geqslant q^2 \geqslant 3.15$ and for $2.64 \geqslant q^2 \geqslant 2.43$. Also, though it is not possible to show them in this figure, there are those in small neighbourhoods of $q^2 = 2\ (4m^2-1)/(4m^2-9)$ $(m = 4, \ldots, 20)$, which correspond to even integral p. There are none in small neighbourhoods of odd integral p. To illustrate these, oscillations of increasing amplitude would have to be drawn to the left of $q^2 = 2.4$. To the right of $q^2 = 5.33$, $(A/B) \sin p\pi$ increases monotonically to $\frac{2}{45}$, as given in (5.1 *c*). In the region $0 \leqslant q^2 \leqslant \frac{2}{9}$ it is found that $(A/B) \sin p\pi$ is negative for $0.21 \leqslant q^2 < \frac{2}{9}$ but is positive otherwise. The values $p = 0, 1, 3, 4, 5, 6, 7$ correspond to $q^2 = 0.222, 0, \infty, 4.286, 3, 2.593, 2.4$ respectively and $q^2 \to 2^+$ as $p \to \infty$. We also note, from (4.15) and (5.2), the increasingly large oscillations in ω_i as p goes through integral values and $q^2 \to 2^+$.

The results for $n = 2$ are similar to those for $n = 1$. There are no centre-modes for $q^2 > 4.46$, but they exist for $4.46 \geqslant q^2 \geqslant 2.94$ and $2.54 \geqslant q^2 \geqslant 2.38$, in which intervals $(A/B) \sin p\pi$ is negative. They also exist in small neighbourhoods of $q^2 = 2[(2m+1)^2-4]/[(2m+1)^2-12]$ $(m = 4, \ldots, 20)$, as has been explicitly checked by evaluating L_P. In the lower interval of q^2, namely $0 \leqslant q^2 \leqslant \frac{2}{3}$, centre-modes are found to exist for $0.58 \leqslant q^2 < \frac{2}{3}$. In this case p takes the integral values 0, 1, 2 in the lower interval and those greater than or equal to 4 in the higher.

The results for $n = 3$ are again similar to those for $n = 1$. The maximum value of q^2 for which centre-modes exist is $q^2 = 3.95$. They exist in intervals $3.95 \geqslant q^2 \geqslant 2.79$ and $2.46 \geqslant q^2 \geqslant 2.33$ and also in the neighbourhoods of $q^2 = 2(4m^2-9)/(4m^2-17)$, this result having been specifically verified for $m = 6, \ldots, 20$. In the lower range of q^2, which now contains the integral values 0, 1, 2, 3 of p, the interval of existence is $0.92 \leqslant q^2 < \frac{18}{17}$.

For $n = 4$ the intervals of existence in the neighbourhoods of certain integral p do not seem to occur for values of p greater than 8. This has been checked by evaluating L_P for P from 8 to 40, and it was found that L_P has sign $(-1)^P$, so that $(A/B) \sin p\pi$ is positive. It could be that A/B has two zeros between such values of q^2, but this seems unlikely in view of our previous experience that A/B had at most one zero between integral values of p. Thus for $q^2 > 2$ there were only two intervals in which

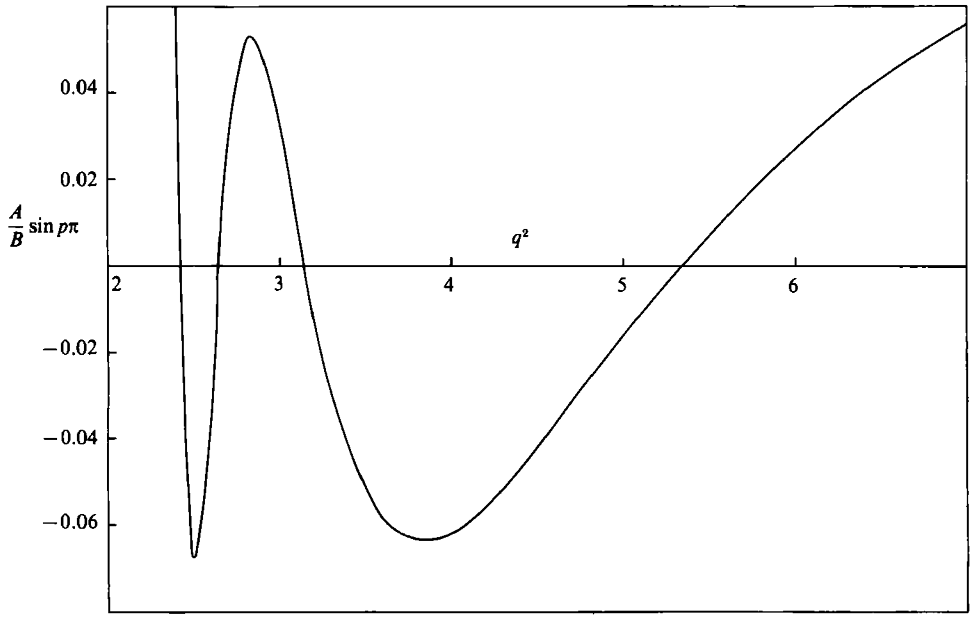

FIGURE. 1. $(A/B)\sin p\pi$ for $n = 1$ and $q^2 > 2$.

centre-modes were found, namely $3.55 \geqslant q^2 \geqslant 2.66$ and $2.40 \geqslant q^2 \geqslant 2.29$. In the lower interval they exist for $1.21 \leqslant q^2 < \frac{4}{3}$.

The disappearance of these intervals of existence in the neighbourhood of $q^2 = 2$ as n increased from 3 to 4 was examined. At $n = 3.99$ there were at least 20 intervals in which $(A/B)\sin p\pi$ was negative, but, by $n = 4$, of these 20 only the two corresponding to the largest values of q^2 remained. The results are illustrated in figure 2. A solid line indicates a region of existence, the length of which was specifically calculated by evaluating $(A/B)\sin p\pi$ for non-integral p by integration of (3.5). The crosses indicate the approximate position of an integral value of p in the neighbourhood of which $(A/B)\sin p\pi < 0$. Between each cross there is a neighbourhood in which the centre-modes do not exist. For $n = 1, 2, 3$ it is suspected that the crosses to the right of $q^2 = 2$ should be infinite in number. By the time n reached the value 5 the main interval extended from $3.21 \geqslant q^2 \geqslant 2.55$, but it proved impossible to compute the length of the second interval to the right of $q^2 = 2$, probably because of the, by now, very small radius of convergence of the series in (3.2) to which the Runge–Kutta solution had to be matched. At $n = 6$ the main interval has $2.84 \geqslant q^2 \geqslant 2.44$, and there are three to the right and two to the left of $q^2 = 2$. At $n = 7$ the main interval has $2.50 \geqslant q^2 \geqslant 2.36$, and there are three to the left and two to the right of $q^2 = 2$. From now on the intervals to the left of $q^2 = 2$ increase in number and those to the right disappear. At $n = 8$ there is one to the right and three to the left; at $n = 9$ there are none to the right and four to the left. For $n = 20$ there are ten to the left of $q^2 = 2$, extending from $1.81 \leqslant q^2 < 1.96$, and for $n = 30$ there are fifteen, extending from $1.87 \leqslant q^2 < 1.98$. Presumably as $n \to \infty$ the number will become infinite and centred on $q^2 = 2^-$.

Before discussing, in §7, the implications of these results, we show that there is no such similar behaviour in the neighbourhood of $\beta = 0$, and indeed there are no centre-modes there.

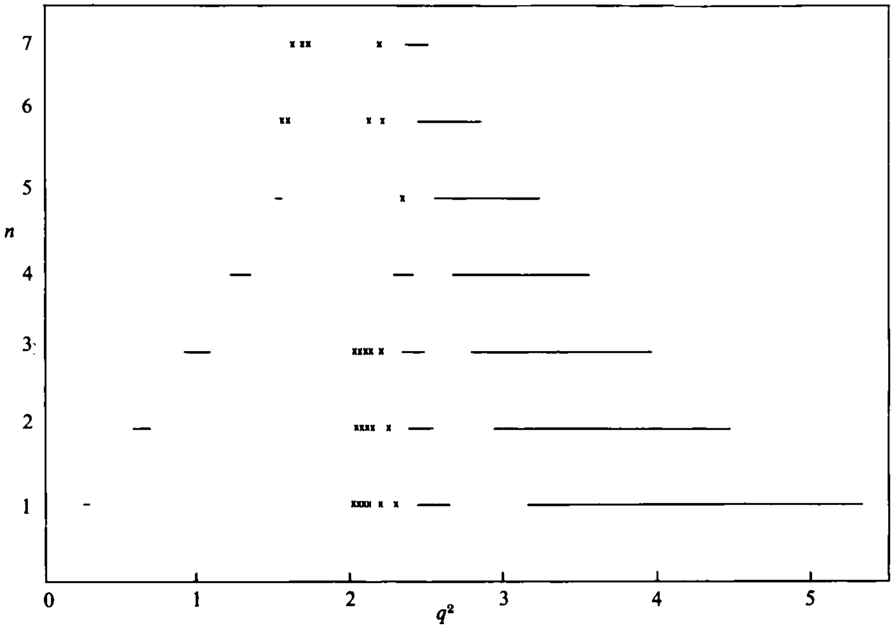

FIGURE 2. Illustration of the regions of existence and non-existence of centre-modes as q^2 varies for various values of n. The solid lines indicate regions of existence, the lengths of which have been evaluated explicitly; the intervals of existence whose lengths were not calculated are represented by a cross that marks one point of the interval.

6. The non-existence of centre-modes in the neighbourhood of $\beta = 0$

When $|\beta| \ll 1$ a discussion analogous to that of §§2–5 for the case $|1-\beta q| \ll 1$ is possible, and we outline it briefly here. In this situation (2.5) and (2.6) become

$$\omega = -nq(1-\tfrac{1}{2}\Gamma), \quad \gamma(r) \approx -\tfrac{1}{2}nq(\Gamma - r^2) \tag{6.1}$$

respectively, where Γ is a small constant to be found, the order of magnitude of which will have $|\Gamma| = O(\beta)$. Instead of (3.1) for the region $r = O(1)$, we now obtain

$$r\frac{\mathrm{d}}{\mathrm{d}r}(ry') - \left(n^2 + \frac{4r^4\,\mathrm{e}^{-r^2}}{r^2 - 1 + \mathrm{e}^{-r^2}}\right) y = 0, \tag{6.2}$$

on setting $\beta = 0 = \Gamma$ in (2.1). The outer solution is thus, to leading order, independent of q, and is again of the form (3.2), but this time with $p = p_0$, where

$$p_0^2 = n^2 + 8. \tag{6.3}$$

As before, it is the ratio A_0/B_0 for the solution of (6.2) that decays as $r \to \infty$ that is required for the matching procedure.

For the inner region where $r = O|\Gamma|$ the equation corresponding to (4.2) is, with $r^2 = \Gamma s$ and $s = O(1)$,

$$s^2\frac{\mathrm{d}^2\tilde{Y}}{\mathrm{d}s^2} + s\frac{\mathrm{d}\tilde{Y}}{\mathrm{d}s} + \left(-\frac{n^2}{4} + \frac{2s}{1-s} - \frac{4\beta s}{\Gamma q(1-s)^2}\right)\tilde{Y} = 0, \tag{6.4}$$

with solution
$$\tilde{Y}(s) = \frac{s^{\frac{1}{2}n}}{(1-s)^{\mu}} F(a, b, n+1, s). \tag{6.5}$$

n	A_0/B_0	p_0^2
2	-0.4011	12
3	$+0.2812$	17
4	$+0.0069$	24
5	-0.00035	33

TABLE 1. The values of A_0/B_0 for $2 \leqslant n \leqslant 5$

Here now $$\mu(\mu+1) = 4\beta/\Gamma q \tag{6.6}$$

and a and b are again given by (4.5), but with $p = p_0$ as in (6.3). The matching is as before, and the requirement for consistency is also (4.16), i.e.

$$\frac{A_0}{B_0}\sin p_0\pi < 0. \tag{6.7}$$

Since neither A_0/B_0 nor p_0 depend on q, the condition (6.7) may be checked by varying n alone. It is easy to show that it is not satisfied when $n = 1$ ($p_0 = 3$). For $p_0 \approx 3$ it may be shown from (6.2) that

$$\frac{A_0}{B_0} = -\frac{2}{45}\frac{1}{p_0-3}+O(1), \tag{6.8}$$

is following as in §3 that A_0/B_0 has a simple pole at each integral value of p_0. Thus as $p_0 \to 3$

$$\frac{A_0}{B_0}\sin p_0\pi \to \frac{2\pi}{45} > 0. \tag{6.9}$$

For higher values of n the solution of (6.2) and evaluation of A_0/B_0 must be numerical as outlined in §3. For $n = 2, \ldots, 5$ these are listed in table 1, from which it can be seen that (6.7) is not satisfied for such values of n.

One might expect inconsistency for higher values of n also. As an indication of accuracy, we report the value $A_0/B_0 = -133.11$ obtained when $n = 1.001$, a result that is in good agreement with (6.8) since $p_0-3 \approx \frac{1}{3}(n-1)$ when $n \approx 1$.

The conclusion to be drawn from this section is negative in the sense that the discussion shows that $\beta = 0$ is not the lower neutral point, at least for centre-modes of the type considered here. Indeed the published numerical work gives little indication that it is, though S. Leibovich (private communication) reports a tendency for ω_i to remain non-zero near $\beta = \frac{1}{2}q$, the stronger candidate for the lower neutral point when $q^2 < 2$. It therefore seemed worthwhile to investigate the status of $\beta = 0$ in terms of the analysis considered here, and the conclusion is that there are no centre-modes in this neighbourhood, at least for $1 \leqslant n \leqslant 5$.

7. Discussion

The rather bizarre results of the preceding paragraphs may offer a partial explanation of the difficulties experienced by numerical analysts in computing the neutral or near-neutral modes for the vortical flow given by (1.1). It has been shown that centre-modes exist in the neighbourhood of $\beta = q^{-1}$ in distinct and discrete intervals of q^2 that also depend strongly on the value of the azimuthal wavenumber

n. For all these unstable modes the growth rate is very small. Since (4.15) shows that the growth rate is also proportional to n, the extension of figure 2 to $n = 0$, which gives a result similar to the case $n = 1$, is not in contradiction with the fact that the flow is known (see Stewartson & Leibovich 1985) to be stable to axisymmetric disturbances for $q^2 > 0.16$.

The interpretation of the fact that there are intervals of q^2 for which the present centre-modes do not exist is most likely not that the flow is stable for such values of q^2 but that, in these cases, either the modes are of a type different from the centre-modes discussed here, or that $\beta = q^{-1}$ is not for them a neutral point. Stewartson & Leibovich (1985), in their investigation of marginal separation at large values of n, mentioned here in §1, show that if $q^2 > 2$ the neutral point is not identically q^{-1}. As n increases, the conclusions of the present study are not inconsistent with those of Leibovich & Stewartson (1983) and Stewartson & Capell (1985), who predict stability for $q^2 > 2$ in the limit $n \to \infty$. In addition, for larger values of n we seem to be obtaining centre-modes in the limit $q^2 \to 2^-$ only; the ring-modes of Stewartson & Capell also become centre-modes in this limit.

A point of note is that we have obtained instabilities at greater values of q^2 (≈ 5.33) than have hitherto been reported. Lessen *et al.* (1974) report $q^2 \approx 2.25$, though they do state that instability at higher values of the swirl parameter was found by Bergman (1969) for a swirling flow with somewhat similar velocity profiles.

This research, the major part of which was completed before the death of Keith Stewartson in May 1983, was supported by the Fluid Mechanics Programme of the National Science Foundation under Grant MEA8306713. Both authors benefited greatly from the hospitality of, and stimulating discussions with, Professor S. Leibovich at Cornell University.

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